

DIAMETER TWO PROPERTIES IN JAMES SPACES

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ABSTRACT. We study the diameter two properties in the spaces JH , JT_∞ and JH_∞ . We show that the topological dual space of the previous Banach spaces fails every diameter two property. However, we prove that JH and JH_∞ satisfy the strong diameter two property, and so the dual norm of these spaces is octahedral. Also we find a closed hyperplane M of JH_∞ whose topological dual space enjoys the w^* -strong diameter two property and also M and M^* have an octahedral norm.

1. INTRODUCTION.

The well known Radon-Nikodym property (RNP) in Banach spaces is characterized by the existence of slices with arbitrarily small diameter in every non empty, closed and bounded subset of the space. In last years a new topic has emerged, extremely different to the RNP: the diameter two properties.

We say that a Banach space has the slice diameter two property (slice-D2P), respectively diameter two property (D2P), strong diameter two property (SD2P) if every slice (respectively non-empty weakly open set, convex combination of slices) in its unit ball has diameter two. For a dual Banach space, we say that it has the w^* -slice diameter two property (respectively w^* -diameter two property, w^* -strong diameter two property) if every w^* slice (respectively non-empty w^* open set, convex combination of w^* -slices) in its unit ball has diameter two. It is known that the six above properties are extremely different as proved in [3].

It is known a wide class of Banach spaces enjoying some of the previous properties as infinite-dimensional uniform algebras [15], infinite-dimensional C^* -algebras [1], non-reflexive M -embedded spaces [10], Banach spaces with the Daugavet property [17], etc, which shows that these properties have strong links with another well known properties as containing an isomorphic copy of ℓ_1 [2] or Daugavet property.

In spite of the wide study about the size of slices, non-empty weakly open subsets and convex combination of slices in the unit ball of several Banach spaces, non-classical Banach spaces have not been deeply studied yet.

Probably the origin of non-classical Banach spaces was in [11], where the space J (James space) is constructed in order to provide an example of a non-reflexive Banach space which fails to contain an isomorphic copy of c_0 or

ℓ_1 . A year later [12], James went further and modified the definition of the norm in order to show that J and J^{**} are isometrically isomorphic, in spite of the non-reflexivity of J (see [6] or [14] for background about J space). It is known that J and J^* have the RNP as dual and separable Banach spaces.

After the construction of J space, James constructed in [13] the JT space (James tree space), exhibiting an example of separable Banach space whose topological dual space is not separable and that does not contain any isomorphic copy of ℓ_1 , giving a negative answer to a conjecture of Stephan Banach (again we refer to [6] or [14] for background in JT space). It is known that JT satisfies the RNP and that B , the predual of JT , does not have the RNP [6]. The fact that B fails the RNP is far to prove that B has the slice-D2P. Indeed, in [16, Theorem 5.1] it is proved the existence of a constant $0 < \beta < 2$ such that every closed and convex subset of the unit ball of B has a slice of diameter less than or equal to β , being the first non-classical Banach space whose size of the slices of its unit ball is studied. In fact, it is conjectured in [16, Remark 5.2] that the above constant β could be, at most, $\sqrt{2}$.

Motivated by the analysis of B , the aim of this note is to study the slices of the unit ball of another exotic spaces. Indeed, in section 2 we focus our attention in JT_∞ space, showing that JT_∞^* fails the w^* -slice diameter two property, and so every diameter two property. Note that JT_∞ is a separable dual space and then satisfies RNP. Then the space B_∞ , the predual space of JT_∞ , fails every diameter two property. In fact, we prove that the inf of the diameters of slices in the unit ball of B_∞ is, at most, $\sqrt{2}$. The same fact also holds for the predual space B of JT , which proves that the suspect in [16, Remark 5.2] holds for the unit ball. In section 3 we prove that the unit ball of JH has a Fréchet differentiability point and, as a consequence, the unit ball of JH^* contains w^* -slices of arbitrarily small diameter, failing each property of diameter two. However it is proved in this section that JH in fact has the strong diameter two property, and so JH satisfies every diameter two property. As a consequence the norm in the dual space JH^* is octahedral. In section 4 we introduce the JH_∞ space, a Banach space not isomorphic to JH , showing that unit ball of JH_∞^* has w^* slices of diameter strictly less than 2. Also one can prove that JH_∞ has the strong diameter two property and so an octahedral dual norm. Finally, in section 5, we find a closed hyperplane M of JH_∞ such that M^* satisfies the w^* -strong diameter two property and also M and M^* have an octahedral norm.

We pass now to introduce some notation. We consider real Banach spaces. B_X and S_X stand for the closed unit ball and the unit sphere of the Banach space X . We denote by X^* the topological dual space of X . For a slice of a bounded subset $A \subseteq X$ we mean the set

$$S(A, x^*, \alpha) := \{x \in A / x^*(x) > \sup x(A) - \alpha\}$$

for $x^* \in X^*$ and $0 < \alpha$. By a w^* -slice of a bounded subset $B \subseteq X^*$ we mean the set

$$S(B, x, \alpha) := \{x^* \in B \mid x^*(x) > \sup x(B) - \alpha\}$$

for $x \in X$ and $0 < \alpha$.

Recall that the norm of a Banach space X is octahedral (see [5]) if for every $\varepsilon > 0$ and for every finite-dimensional subspace Y of X there is $x \in S_X$ such that

$$\|\lambda x + y\| > (1 - \varepsilon)(|\lambda| + \|y\|)$$

for every $y \in Y$ and for every scalar λ . We remark that the norm of a Banach space X is octahedral if, and only if, X^* satisfies the w^* -strong diameter two property and, dually, the norm of X^* is octahedral if, and only if, X satisfies the strong diameter two property (see [2]).

Also we recall that a Banach space X has the Daugavet property if the equation

$$(1.1) \quad \|T + I\| = 1 + \|T\|$$

for every rank one, linear and bounded operator on X , where I denotes the identity operator. X is said to have the almost Daugavet property if there is some norming subspace Y of X^* such that the equation 1.1 holds for every rank one operator T given by $T = x \otimes y^*$ for $x \in X$ and $y^* \in Y$. It is known [9] that, for a separable Banach space, having octahedral norm and satisfying the almost Daugavet property are equivalent. Also this is equivalent to say that X^* has the w^* -strong diameter two property, as can be deduced from the comments in the above paragraph. These facts will be used freely below.

The following known result, see Lemma 2.1 and Proposition 3.1 in [4], will be useful in order to estimate the inf of the diameters of w^* -slices in dual spaces.

Theorem 1.1. *Let X be a Banach space and assume that $A \subseteq X^*$ satisfies that $B_{X^*} = \overline{\text{co}}^{w^*}(A)$. If $x \in S_X$, then*

$$\inf_{\alpha > 0} \text{diam}(S(A, x, \alpha)) = \inf_{\alpha > 0} \text{diam}(S(B_{X^*}, x, \alpha)).$$

2. THE SPACE JT_∞ .

We shall begin with the construction of JT_∞ space. Define

$$T := \{(\alpha_1, \dots, \alpha_k) \mid k \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{N}\} \cup \{\emptyset\}.$$

Given $(\alpha_1, \dots, \alpha_k), (\beta_1, \dots, \beta_p) \in T$ we say that

$$(\alpha_1, \dots, \alpha_k) \leq (\beta_1, \dots, \beta_p) \Leftrightarrow \begin{cases} k \leq p \\ \alpha_i = \beta_i \quad \forall i \in \{1, \dots, k\} \end{cases}.$$

Last relation defines a partial order defining $|(\alpha_1, \dots, \alpha_n)| = n$ and $|\emptyset| = 0$.

By a segment we mean a subset $S \subseteq T$ totally ordered and finite.
Given $x : T \rightarrow \mathbb{R}$ we consider

$$\|x\| = \sup \left(\sum_{i=1}^n \left(\sum_{t \in S_i} x(t) \right)^2 \right)^{\frac{1}{2}},$$

where the sup is taken over all families $\{S_1, \dots, S_n\}$ of disjoint segments in T .

Then JT_∞ is defined as the completion of the space of finitely nonzero functions defined on T in the above norm. Given $\alpha \in T$ define

$$e_\alpha(\beta) := \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then $\{e_\alpha\}_{\alpha \in T}$ defines a Schauder basis on JT_∞ . Denote by $\{e_\alpha^*\}_{\alpha \in T}$ the biorthogonal sequence and let $B_\infty := \overline{\text{span}}\{e_t^* / t \in T\}$.

The space JT_∞ was introduced in [7], where it is proved that B_∞ , being the predual of JT_∞ , fails the Radon-Nikodym property. Furthermore, every infinite subspace of JT_∞ contains an isomorphic copy of ℓ_2 and so JT_∞ does not contain isomorphic copies of ℓ_1 .

Now we pass to talk about the size of slices in $B_{JT_\infty^*}$. As in [16], we define a molecule as a functional of the form

$$x^* := \sum_{i=1}^n \lambda_i f_{S_i}$$

for S_1, \dots, S_n disjoint segments in T and $\sum_{i=1}^n \lambda_i^2 \leq 1$, where

$$f_S(x) = \sum_{t \in S} x(t)$$

for $S \subseteq T$ a segment.

Denote by M the set of molecules in JT_∞^* and note that $M \subseteq B_{JT_\infty^*}$.

We shall begin with the following Lemma, which shows that M is a norming subset of $B_{JT_\infty^*}$.

Lemma 2.1. *M is a norming subset of $B_{JT_\infty^*}$. As a consequence*

$$(2.1) \quad B_{JT_\infty^*} = \overline{\text{co}}^{w^*}(M).$$

Proof. Let $x \in S_{JT_\infty}$ a finitely nonzero function defined on T . Pick an arbitrary $0 < \varepsilon < 1$ and take $0 < \delta < 1$ such that $(1 - \delta)^2 > 1 - \varepsilon$. By the definition of the norm in JT_∞ we deduce that there exist S_1, \dots, S_n disjoint segments in T such that

$$\left(\sum_{i=1}^n f_{S_i}(x)^2 \right)^{\frac{1}{2}} > 1 - \delta.$$

For every $i \in \{1, \dots, n\}$ define $\lambda_i := f_{S_i}(x)$ and note that by the definition of the norm in JT_∞ it is clear that $\sum_{i=1}^n \lambda_i^2 \leq 1$. Moreover, in view of last inequality, we have

$$\sum_{i=1}^n \lambda_i f_{S_i}(x) = \sum_{i=1}^n f_{S_i}(x)^2 > (1 - \delta)^2 > 1 - \varepsilon.$$

As a consequence we can find in M elements whose evaluation in x is as close to $\|x\|$ as desired. Hence M is a norming subset of $B_{JH_\infty^*}$.

From a separation argument we get now that $B_{JT_\infty^*} = \overline{co}^{w^*}(M)$.

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Using previous lemma, we will prove that there exist w^* -slices in $B_{JT_\infty^*}$ with diameter strictly less than 2.

Theorem 2.2. *There exists $x \in S_{JT_\infty}$ such that*

$$\inf_{\alpha > 0} \text{diam } S(B_{JT_\infty^*}, x, \alpha) \leq \sqrt{2}.$$

Proof. Let $0 < \varepsilon < 1/2$. Pick $0 < \delta < \min\{\varepsilon, 2\varepsilon(1 - \varepsilon)\}$ and $0 < \alpha < 1/2$ such that $(1 - \alpha)^2 > 1 - \delta$. Define

$$x := (1 - \varepsilon)e_\emptyset + \varepsilon e_{(1)} \in S_{JT_\infty}.$$

Consider $S := S(M, x, \alpha)$. Take $\sum_{i=1}^n \lambda_i f_{S_i}, \sum_{j=1}^m \mu_j f_{T_j} \in S$.

In view of the form of x we can ensure the existence of $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ such that $\{\emptyset, (1)\} \subseteq S_i \cap T_j$. Indeed, it is clear that $(\cup_{i=1}^n S_i) \cap \{\emptyset, (1)\} \neq \emptyset$, since $\sum_{i=1}^n \lambda_i f_{S_i} \in S$. Now it is not possible that $(\cup_{i=1}^n S_i) \cap \{\emptyset, (1)\} = \{\emptyset\}$ nor $(\cup_{i=1}^n S_i) \cap \{\emptyset, (1)\} = \{(1)\}$, since $0 < \varepsilon < 1/2$, $0 < \alpha < 1/2$, $\sum_{i=1}^n \lambda_i^2 \leq 1$ and $\sum_{i=1}^n \lambda_i f_{S_i} \in S$. Finally, it is not possible that there exist $i \neq j$ such that $\{\emptyset\} \in S_i$ and $\{(1)\} \in S_j$, since if this is the case, we have that $(1 - \varepsilon)\lambda_i + \varepsilon\lambda_j > 1 - \alpha$. Hence

$$(1 - \alpha)^2 < ((1 - \varepsilon)^2 + \varepsilon^2)(\lambda_i^2 + \lambda_j^2) \leq (1 - \varepsilon)^2 + \varepsilon^2$$

and thus, using the conditions on α, δ and ε , we get

$$1 - 2(\varepsilon(1 - \varepsilon)) < 1 - \delta < (1 - \varepsilon)^2 + \varepsilon^2,$$

which is a contradiction. This proves the existence of i such that $\{\emptyset, (1)\} \subseteq S_i$. The same argument proves the existence of j such that $\{\emptyset, (1)\} \subseteq T_j$. Of course, we assume without loss of generality that $i = j = 1$. Now

$$\sum_{i=1}^n \lambda_i f_{S_i}(x) = \lambda_1(1 - \varepsilon + \varepsilon) = \lambda_1 > 1 - \alpha \Rightarrow \lambda_1^2 > (1 - \alpha)^2 > 1 - \delta.$$

As $\sum_{i=1}^n \lambda_i^2 \leq 1$ then $\sum_{i=2}^n \lambda_i^2 < \delta$. By a similar argument $\mu_1^2 > 1 - \delta$ and hence $\sum_{j=2}^m \mu_j^2 < \delta$.

In order to estimate $\left\| \sum_{i=1}^n \lambda_i f_{S_i} - \sum_{j=1}^m \mu_j f_{T_j} \right\|$ pick $y \in S_{JT_\infty}$. Hence

$$\left| \left(\sum_{i=1}^n \lambda_i f_{S_i} - \sum_{j=1}^m \mu_j f_{T_j} \right) (y) \right| \leq \underbrace{|\lambda_1 f_{S_1}(y) - \mu_1 f_{T_1}(y)|}_A + \underbrace{\left| \sum_{i=2}^n \lambda_i f_{S_i}(y) - \sum_{j=2}^m \mu_j f_{T_j}(y) \right|}_B.$$

We shall begin estimating B . In view of Hölder's inequality we have

$$\begin{aligned} B &\leq \sum_{i=2}^n |\lambda_i| |f_{S_i}(y)| + \sum_{j=2}^m |\mu_j| |f_{T_j}(y)| \leq \\ &\leq \left(\sum_{i=2}^n \lambda_i^2 + \sum_{j=2}^m \mu_j^2 \right)^{\frac{1}{2}} \left(\sum_{i=2}^n f_{S_i}(y)^2 + \sum_{j=2}^m f_{T_j}(y)^2 \right)^{\frac{1}{2}} \leq (2\delta)^{\frac{1}{2}} 2^{\frac{1}{2}} = 2\sqrt{\delta} \end{aligned}$$

because $\sum_{i=2}^n f_{S_i}(y)^2 \leq \|y\|^2 = 1$, $\sum_{j=2}^m f_{T_j}(y)^2 \leq 1$ due to the disjointness of $\{S_2, \dots, S_n\}$ and $\{T_2, \dots, T_m\}$. So $B \leq 2\sqrt{\delta}$. Now we will estimate A :

$$A \leq |\lambda_1 - \mu_1| |f_{T_1 \cap S_1}(y)| + |\lambda_1| |f_{S_1 \setminus T_1}(y)| + |\mu_1| |f_{T_1 \setminus S_1}(y)|.$$

As $1 \geq \lambda_1 > 1 - \alpha$ and $1 \geq \mu_1 > 1 - \alpha$ then $|\lambda_1 - \mu_1| < \alpha$. Hence

$$A \leq \alpha \|f_{T_1 \cap S_1}\| \|y\| + |\lambda_1| |f_{S_1 \setminus T_1}(y)| + |\mu_1| |f_{T_1 \setminus S_1}(y)| =$$

$$\alpha + |\lambda_1| |f_{T_1 \setminus S_1}(y)| + |\mu_1| |f_{T_1 \setminus S_1}(y)| \leq \alpha + |f_{S_1 \setminus T_1}(y)| + |f_{T_1 \setminus S_1}(y)|.$$

Again applying Hölder's inequality we have

$$A \leq \alpha + \sqrt{2} (f_{S_1 \setminus T_1}(y)^2 + f_{T_1 \setminus S_1}(y)^2)^{\frac{1}{2}}.$$

As $\{S_1 \setminus T_1, T_1 \setminus S_1\} \subseteq T$ is a family of disjoint segments we have that $f_{S_1 \setminus T_1}(y)^2 + f_{T_1 \setminus S_1}(y)^2 \leq \|x\|^2 = 1$. Hence

$$A \leq \alpha + \sqrt{2}.$$

Summarizing

$$\left| \left(\sum_{i=1}^n \lambda_i f_{S_i} - \sum_{j=1}^m \mu_j f_{T_j} \right) (y) \right| \leq \alpha + \sqrt{2} + 2\sqrt{\delta}.$$

From the arbitrariness of $y \in S_{JT_\infty}$ we have that

$$\left\| \sum_{i=1}^n \lambda_i f_{S_i} - \sum_{j=1}^m \mu_j f_{T_j} \right\| = \sup_{y \in S_{JT_\infty}} \left| \left(\sum_{i=1}^n \lambda_i f_{S_i} - \sum_{j=1}^m \mu_j f_{T_j} \right) (y) \right| \leq \sqrt{2} + \alpha + 2\sqrt{\delta}.$$

Hence

$$\text{diam}(S) \leq \sqrt{2} + \alpha + 2\sqrt{\delta}.$$

So

$$\inf_{\alpha > 0} \text{diam}(S(M, x, \alpha)) \leq \sqrt{2} + 2\sqrt{\delta}.$$

Since $0 < \delta < \varepsilon$ was arbitrary we deduce that

$$\inf_{\alpha > 0} \text{diam}(S(M, x, \alpha)) \leq \sqrt{2}.$$

In view of Lemma 2.1, Theorem 1.1 applies and

$$\inf_{\alpha > 0} \text{diam}(S(B_{JT_\infty^*}, x, \alpha)),$$

so we are done.

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In view of previous theorem, for each $0 < \varepsilon < 2 - \sqrt{2}$ we can find S a w^* -slice in $B_{JT_\infty^*}$ such that $\text{diam}(S) < \sqrt{2} + \varepsilon$. In particular, JT_∞^* fails to have the w^* -slice diameter two property and hence B_∞ fails every diameter two property, since the inf of the diameters of slices in the unit ball of B_∞ agrees with the inf of the diameters of w^* -slices in the unit ball of JT_∞^* . In fact, this inf is, at most, $\sqrt{2}$. Also, it is possible obtaining the same result for the space B , the predual of JT , with the above proof, which shows that the conjecture in [16] that the inf of diameters of slices in the unit ball in B is, at most, $\sqrt{2}$ holds.

3. THE SPACE JH .

We shall begin with the construction of JH space. Following [6] we denote by

$$T := \{(n, i) / 0 \leq n < \infty, 0 \leq i < 2^n\}$$

the diadic tree. We say that $(n+1, 2i)$ and $(n+1, 2i+1)$ are offspring of (n, i) for every $(n, i) \in T$. A segment will be a non-empty finite sequence

$$S = \{t_1, \dots, t_n\}$$

such that t_{j+1} is an offspring of t_j for every $j \in \{1, \dots, n-1\}$.

Now we are ready to define a partial order in T : given $t_1, t_2 \in T$ we say that $t_1 < t_2$ if, and only if, $t_1 \neq t_2$ and there exists a segment such that t_1 is the first element of the segment and t_2 is the last element on it.

The set

$$\{(n, i) / 0 \leq i < 2^n\}$$

is called the n -th level of T for every $0 \leq n < \infty$.

Given $n, m \in \mathbb{N}, n \leq m$ we will say that a subset $S \subseteq T$ is an $n - m$ segment if

- For every $n \leq k \leq m$ there exists an only element in S which is in the k -th level of T ,
- If $(p, i), (q, j) \in S$ and $p < q$ then $(p, i) < (q, j)$ (in other words, S is a totally ordered subset of T).

Given $x : T \rightarrow \mathbb{R}$ and $S \subseteq T$ a segment in T , we define

$$f_S(x) := \sum_{t \in S} x(t).$$

Note that the above sum is well defined because S is finite.

Given $\{S_1, \dots, S_n\}$ a family of segments in T we say that are admissible if:

- i) There exist $p \leq q$ natural numbers such that S_i is an $p - q$ segment for every $i \in \{1, \dots, n\}$.
- ii) $S_i \cap S_j = \emptyset$ whenever $i \neq j$.

Given $x : T \rightarrow \mathbb{R}$ a finitely nonzero function we define

$$\|x\| := \max \sum_{i=1}^n |f_{S_i}(x)| = \max \sum_{i=1}^n \left| \sum_{t \in S_i} x(t) \right|,$$

where the maximum is taken over all families S_1, \dots, S_n of admissible segments in T .

Now define JH as the completion of the space of finitely nonzero functions on T in the above norm.

Given $t \in T$ we define $e_t \in JH$ by the equation

$$e_t(s) := \begin{cases} 1 & \text{if } t = s \\ 0 & \text{otherwise} \end{cases}.$$

Then $\{e_t\}_{t \in T}$ defines a Schauder basis on JH .

JH space was introduced by J.Hagler in [8]. It is proved in that paper that JH is a separable Banach space such that JH^* is not separable and that every infinite-dimensional subspace of JH contains an isomorphic copy of c_0 . In particular JH contains an isomorphic copy of c_0 , so it can not be a dual space [14, Proposition 2.e.8].

Lemma 3.1. *Let $x : T \longrightarrow \mathbb{R}$ a finitely non-zero function and $n \in \mathbb{N} \setminus \{1\}$ such that*

$$\|x\| \leq 1 - \frac{1}{n}.$$

Pick $a \in T$ such that $\text{lev}(a) > \max_{t \in \text{supp}(x)} \text{lev}(t)$. Let $\ell \in \mathbb{N}$ big enough such that there exists $t_1, \dots, t_n \in T$ such that

- $\text{lev}(t_i) = \ell$ for each i .
- $a < t_i$ for all i .

If we define $y : T \longrightarrow \mathbb{R}$ such that

$$y(t) := \begin{cases} x(t) & t \in \text{supp}(x) \\ \mu_i \frac{1}{n} & t = t_i \text{ } i \in \{1, \dots, n\}, \mu_i \in \{-1, 1\} \\ 0 & \text{otherwise} \end{cases},$$

then $\|y\| \leq 1$.

Proof. Let $\{S_1, \dots, S_k\}$ be a family of admissible segments in T , $\lambda_1, \dots, \lambda_k \in \{-1, 1\}$ and define

$$x^* := \sum_{i=1}^k \lambda_i f_{S_i}.$$

In order to prove that $\|y\| \leq 1$ we have to prove that $x^*(y) \leq 1$, following the definition of the norm in JH .

By cases:

$$(1) \bigcup_{i=1}^k S_i \cap \{t_1, \dots, t_n\} = \emptyset.$$

In this case we have, in view of the definition of y that

$$x^*(y) = x^*(x) \leq \|x\| \leq 1 - \frac{1}{n}$$

by hypothesis.

$$(2) \bigcup_{i=1}^k S_i \cap \{t_1, \dots, t_n\} \neq \emptyset \text{ but } \bigcup_{i=1}^k S_i \cap \text{supp}(x) = \emptyset.$$

In this case we have

$$x^*(y) = \sum_{i=1}^k \lambda_i f_{S_i}(y) \leq \sum_{i=1}^n \frac{1}{n} = 1$$

$$(3) \bigcup_{i=1}^k S_i \cap \{t_1, \dots, t_n\} \neq \emptyset \text{ and } \bigcup_{i=1}^k S_i \cap \text{supp}(x) \neq \emptyset.$$

Finally, in this case we have that there exists one only $i \in \{1, \dots, n\}$

such that $a \in S_i$ (otherwise $\bigcup_{i=1}^k S_i \cap \{t_1, \dots, t_n\} = \emptyset$ in view of the

order defined on T). We can assume, without of generality, that $i = 1$. If S_j is a $p - q$ segment, we can write

$$S_j := T_j \cup R_j$$

where T_j is a $p - (\ell - 1)$ segment and R_j is a $\ell - q$ segment for each $j \in \{1, \dots, k\}$.

In view of the disjointness of S_1, \dots, S_k we have for each $j \in \{2, \dots, n\}$ that $S_j \cap \{t_1, \dots, t_n\} = \emptyset$. In addition, as $\ell > \max_{t \in \text{supp}(x)} \text{lev}(t)$ we deduce that

$$f_{R_i}(y) = 0 \quad \forall i \in \{2, \dots, n\}.$$

Hence

$$x^*(y) = \sum_{i=1}^k \lambda_i f_{T_i}(y) + \lambda_1 f_{R_1}(y).$$

Now we have that $\{T_1, \dots, T_k\}$ is a family of admissible segments on T . Hence

$$x^*(y) \leq \|x\| + \lambda_1 f_{R_1}(x) \leq 1 - \frac{1}{n} + f_{R_1}(y).$$

Now, as $\{t_1, \dots, t_n\}$ are incomparable nodes on T at the same level we have that $\{t_1, \dots, t_n\} \cap R_1$ has one element. Hence

$$x^*(y) \leq 1 - \frac{1}{n} + f_{R_1}(y) \leq 1 - \frac{1}{n} + \frac{1}{n} = 1.$$

By the previous discussion we deduce that $\|y\| \leq 1$ as desired.

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Theorem 3.2. *JH has the strong diameter two property (and so the norm of JH^* is octahedral).*

Proof. Let $C := \sum_{i=1}^n \lambda_i S(B_{JH}, x_i^*, \alpha)$ a convex combination of slices on B_{JH} . Let prove that $\text{diam}(C) = 2$.

To this aim pick $x_i : T \rightarrow \mathbb{R}$ a finitely non-zero supported function on T such that $\|x_i\| < 1$ and

$$x_i^*(x_i) > 1 - \alpha,$$

for each $i \in \{1, \dots, n\}$. For each $i \in \{1, \dots, n\}$ we can find $a_i \in \text{supp}(x_i)$ such that $\text{lev}(a_i) = \max_{t \in \text{supp}(x_i)} \text{lev}(t)$.

As $\|x_i\| < 1$ for each $i \in \{1, \dots, n\}$ we can find $m \in \mathbb{N}$ such that $\|x_i\| \leq 1 - \frac{1}{m}$ for each $i \in \{1, \dots, n\}$. Now we can find $a \in T$ such that $\text{lev}(a) > \max_{1 \leq i \leq n} \text{lev}(a_i)$, $k > \max_{1 \leq i \leq n} \text{lev}(a_i)$ big enough and $\{t_1^i, \dots, t_{2n}^i\}$ a family of nodes

on T at level k such that $a < t_p^i$ for each $i \in \{1, \dots, n\}, p \in \{1, \dots, 2m\}$ and that

$$t_p^i \neq t_q^j \text{ if } i \neq j \text{ or } p \neq q.$$

In other words, last condition guaranties that $\{t_p^i / i \in \{1, \dots, n\}, j \in \{1, \dots, 2m\}\}$ is a family of nodes pairwise different nodes at level k which are bigger than a .

For each $i \in \{1, \dots, n\}$ we define $y_i, z_i : T \longrightarrow \mathbb{R}$ a finitely non-zero function on T as follows

$$y_i(t) := \begin{cases} x_i(t) & \text{if } t \in \text{supp}(x_i) \\ \text{sign}\left(x_i^*\left(e_{t_p^i}\right)\right) \frac{1}{m} & t = t_p^i \text{ } p \in \{1, \dots, m\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$z_i(t) := \begin{cases} x_i(t) & \text{if } t \in \text{supp}(x_i) \\ \text{sign}\left(x_i^*\left(e_{t_p^i}\right)\right) \frac{1}{m} & t = t_p^i \text{ } p \in \{m+1, \dots, 2m\} \\ 0 & \text{otherwise} \end{cases}.$$

In view of Lemma 3.1 we have that $\|y_i\| \leq 1$ and $\|z_i\| \leq 1$.

Let prove that, in fact, $y_i, z_i \in S(B_{JH}, x_i^*, \alpha)$ for each $i \in \{1, \dots, n\}$. To this aim pick $i \in \{1, \dots, n\}$ and we shall prove that $y_i \in S(B_{JH}, x_i^*, \alpha)$, being the case of z_i similar. Using the linearity of x_i^* we have

$$\begin{aligned} x_i^*(y_i) &= x_i^*(x_i) + \sum_{p=1}^m \frac{1}{m} \text{sign}\left(x_i^*\left(e_{t_p^i}\right)\right) x_i^*\left(e_{t_p^i}\right) = \\ &= x_i^*(x_i) + \sum_{p=1}^m \frac{1}{m} \left| x_i^*\left(e_{t_p^i}\right) \right| \geq x_i^*(x_i) > 1 - \alpha. \end{aligned}$$

Hence $\sum_{i=1}^n \lambda_i y_i, \sum_{i=1}^n \lambda_i z_i \in C$. Then

$$\text{diam}(C) \geq \left\| \sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i z_i \right\|.$$

Now we shall prove that $\|\sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i z_i\| = 2$. To this aim check that $\{\{t_p^i\} / i \in \{1, \dots, n\}, p \in \{1, \dots, 2m\}\}$ is a family of admissible segments on T . Hence

$$f := \sum_{i=1}^n \sum_{p=1}^m \text{sign}\left(x_i^*\left(e_{t_p^i}\right)\right) f_{\{t_p^i\}} - \sum_{p=m+1}^{2m} \text{sign}\left(x_i^*\left(e_{t_p^i}\right)\right) f_{\{t_p^i\}}$$

is an element on JH^* whose norm is less or equal to one (in view of the definition of the norm in JH). So

$$\begin{aligned}
\left\| \sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i z_i \right\| &\geq f \left(\sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i z_i \right) = \\
&= \sum_{i=1}^n \lambda_i \frac{1}{m} \sum_{p=1}^m \text{sign} \left(x_i^* \left(e_{t_p^i} \right) \right)^2 + \lambda_i \frac{1}{m} \sum_{p=m+1}^{2m} \text{sign} \left(x_i^* \left(e_{t_p^i} \right) \right)^2 = \\
&= 2 \sum_{i=1}^n \lambda_i = 2.
\end{aligned}$$

So $\left\| \sum_{i=1}^n \lambda_i y_i - \sum_{i=1}^n \lambda_i z_i \right\| = 2$, as wanted.

■

We pass now to study the diameter two property on JH^* . Our aim is to prove that B_{JH^*} has w^* -slices with arbitrary small diameter. In fact, we will find $x \in S_{JH}$ such that $\inf_{\alpha > 0} \text{diam}(S(B_{JH^*}, x, \alpha)) = 0$.

If we denote by

$$A := \left\{ \sum_{i=1}^n \lambda_i f_{S_i} \mid \lambda_i \in \{-1, 1\} \text{ and } \{S_1, \dots, S_n\} \text{ is a family of admissible segments in } T \right\},$$

it is clear that $A \subseteq B_{JH^*}$ is a norming subset (by the definition of the norm on JH). Hence

$$\overline{co}^{w^*}(A) = B_{JH^*}$$

by Hahn-Banach theorem.

Now we are ready to show that B_{JH^*} has w^* -slices of arbitrarily small diameter.

Theorem 3.3. *There exists $x \in S_{JH}$ satisfying that*

$$\inf_{\alpha > 0} \text{diam}(S(B_{JH^*}, x, \alpha)) = 0.$$

Proof. Pick $0 < \varepsilon < \frac{1}{4}$ and let

$$x = (1 - \varepsilon)e_{(0,0)} + \varepsilon e_{(1,0)} - \varepsilon e_{(1,1)} - \varepsilon e_{(2,0)} - \varepsilon e_{(2,1)} - \varepsilon e_{(2,2)} + \varepsilon e_{(2,3)}.$$

It is clear that $\|x\| \geq 1$ considering the family of admissible segments $\{(0,0), (1,0)\}$. It can also be checked that if $\{S_1, \dots, S_r\}$ is a family of admissible segments in T which is different of the family $\{(0,0), (1,0)\}$ then

$$\sum_{i=1}^r \left| \sum_{t \in S_i} x(t) \right| \leq \max\{1 - \varepsilon, 4\varepsilon\} < 1.$$

Hence $\|x\| = 1$. Moreover, if we take $\{S_1, \dots, S_r\}$ a family of admissible segment, $\lambda_1, \dots, \lambda_r \in \{-1, 1\}$ such that

$$\sum_{i=1}^r \lambda_i f_{S_i}(x) > 1 - \alpha$$

for $0 < \alpha < \min\{1 - 4\varepsilon, \varepsilon\} < 1$ then $r = 1$, $S_1 = \{(0, 0), (1, 0)\}$ and $\lambda_1 = 1$. So

$$S(A, x, \alpha) = \{f_{\{(0,0),(1,0)\}}\} \Rightarrow \inf_{\alpha > 0} \text{diam}(S(A, x, \alpha)) = 0.$$

Now Theorem 1.1 applies and as a consequence we get that

$$\inf_{\alpha > 0} \text{diam}(S(B_{JH^*}, x, \alpha)) = 0,$$

so we are done. ■

Let's remark that x of the above theorem is a Fréchet differentiability point of B_{JH} , see [5], so as a consequence of the above result we deduce that the unit ball JH^* has denting points.

4. THE SPACE JH_∞ .

We shall begin with the construction of JH_∞ space from the JH space, by a similar process to the construction of JT_∞ space from JT space.

Consider T as in section 2. A segment $S = \{t_1, \dots, t_k\}$ is a $n - m$ segment, for $n \leq m$, if $|t_1| = n$ and $|t_k| = m$.

If $\{S_1, \dots, S_k\}$ is a finite family of segments in T we say that is admissible if

- (1) Exist natural numbers n, m satisfying $n \leq m$ and S_i is a $n - m$ segment for every $i \in \{1, \dots, k\}$.
- (2) $S_i \cap S_j = \emptyset$ if $i \neq j$.

Given $x : T \longrightarrow \mathbb{R}$ a finitely nonzero function we define

$$\|x\| := \sup \sum_{i=1}^k \left| \sum_{t \in S_i} x(t) \right|,$$

where the sup is taken over all families of admissible segments $\{S_1, \dots, S_k\}$ in T .

We define the space JH_∞ as the completion of the space of finitely nonzero functions on T in the above norm.

If $S \subseteq T$ is a segment then we denote by

$$f_S(x) := \sum_{t \in S} x(t).$$

Note that $f_S \in S_{(JH_\infty)^*}$.

Moreover, in view of the definition of the norm we have that given a family of admissible segments $\{S_1, \dots, S_k\}$ then $\sum_{i=1}^k \lambda_i f_{S_i} \in B_{(JH_\infty)^*}$, whenever $\lambda_1, \dots, \lambda_k \in \{-1, 1\}$.

Given $\alpha \in T$ define

$$e_\alpha(\beta) := \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then $\{e_\alpha\}_{\alpha \in T}$ defines a Schauder basis on JH_∞ .

Let us remark that JH_∞ is not isomorphic to JH . Indeed, we know that JH does not contain isomorphic copies of ℓ_1 , however it is enough consider the sequence $\{e_{\alpha_n}\}$, where $\{\alpha_n\}$ is an infinite sequence of immediate successors of the first node in T , to get an isometric copy of the usual basis in ℓ_1 . Furthermore it is clear that JH_∞ contains isometric copies of JH . Now, we can get, as in the previous section for JH , the following result.

Theorem 4.1. *JH_∞ has the strong diameter two property (and so the norm of JH_∞^* is octahedral).*

In order to study diameter two properties in JH_∞^* , next Lemma will help us to estimate the diameter of certain w^* -slices in $B_{JH_\infty^*}$.

Proposition 4.2. *Let R, S be two disjoint segments in T such that are $p-q$ and $p-r$ segments for suitable $p, q, r \in \mathbb{N}, p \leq q \leq r$. Then*

$$\|f_R - f_S\| \leq \frac{5}{3}.$$

Proof. If $r = q$ then $\{S, R\}$ is a family of admissible segments in T . Hence

$$\|f_R - f_S\| = 1 < \frac{5}{3}.$$

Now assume that $q < r$. Then we can find U a $p-q$ segment and V an $(q+1)-r$ segment such that

$$(4.1) \quad U \cup V = R \Rightarrow f_R = f_U + f_V.$$

Let $\alpha \in \mathbb{R}_0^+$ such that $\|f_R - f_S\| = 2 - \alpha$ and $\varepsilon \in \mathbb{R}^+$. Then there exists a finitely nonzero function $x : T \rightarrow \mathbb{R}$, $\|x\| \leq 1$, such that

$$(f_R - f_S)(x) > 2 - \alpha - \varepsilon \Rightarrow f_R(x) > 1 - \alpha - \varepsilon \quad \text{and} \quad f_S(x) < -1 + \alpha + \varepsilon.$$

As U is a $p-q$ segment disjoint with S we have that $\{U, S\}$ is a family of admissible segments. As a consequence $\|f_U - f_S\| \leq 1$. Hence

$$2 - \alpha - \varepsilon < f_R(x) - f_S(x) = (f_U - f_S)(x) + f_V(x) \leq 1 + f_V(x)$$

and so

$$f_V(x) > 1 - \alpha - \varepsilon.$$

Moreover

$$1 \geq f_R(x) = f_U(x) + f_V(x) \geq 1 - \alpha - \varepsilon + f_U(x),$$

hence

$$(4.2) \quad f_U(x) \leq \alpha + \varepsilon.$$

Now, again using that $\{S, U\}$ is a family of admissible segments, we have that $\|f_U + f_S\| \leq 1$. Hence

$$-1 \leq (f_U + f_S)(x) < f_U(x) + (-1 + \alpha + \varepsilon).$$

Then

$$(4.3) \quad f_U(x) > -\alpha - \varepsilon.$$

Combining both (4.2) and (4.3) it follows

$$(4.4) \quad |f_U(x)| \leq \alpha + \varepsilon.$$

Now, as x has finite support, we can find W a $(q+1) - r$ segment such that $S \cup W$ is a $p - r$ segment disjoint with R and we can assume that $x(t) = 0 \ \forall t \in W$. From here, we deduce that $\{R, S \cup W\}$ is a family of admissible segments in T . Hence

$$1 \geq \|x\| \geq |f_R(x)| + |f_{S \cup W}(x)| = |f_U(x) + f_V(x)| + |f_S(x)| \geq$$

$$\geq |f_V(x)| - |f_U(x)| + |f_S(x)| \geq (1 - \alpha - \varepsilon) - (\alpha + \varepsilon) + (1 - \alpha - \varepsilon) = 2 - 3\alpha - 3\varepsilon.$$

From the arbitrariness of ε we deduce that

$$1 \geq 2 - 3\alpha \Rightarrow \alpha \geq \frac{1}{3}.$$

Then $\|f_S - f_T\| = 2 - \alpha \leq 2 - \frac{1}{3} = \frac{5}{3}$, as wanted.

■

Now we can conclude that there are w^* -slices in $B_{JH_\infty}^*$ with diameter strictly less than two. In fact, we can find w^* -slices with diameter less than $\frac{5}{3} + \varepsilon$ for every $0 < \varepsilon < \frac{1}{3}$.

Theorem 4.3. *There exists $x \in S_{JH_\infty}$ such that*

$$\inf_{\alpha > 0} S(B_{JH_\infty}^*, x, \alpha) \leq \frac{5}{3}.$$

Proof. Define

$$A := \left\{ \sum_{i=1}^n \lambda_i f_{S_i} \mid \left\{ S_1, \dots, S_n \right\} \text{ family of admissible segments, } |\lambda_i| = 1 \ i \in \{1, \dots, n\} \right\}.$$

It is clear that $\overline{co}^{w^*}(A) = B_{JH_\infty^*}$ by an easy separation argument. Let $0 < \delta < 1$.

Define $x := (1 - \delta)e_\emptyset + \delta e_{(1)}$. Pick $0 < \alpha < \delta < 1/2$. Then if $\sum_{i=1}^n \lambda_i f_{S_i} \in S(A, x, \alpha)$ we have that $n = 1$, S_1 is a $0 - p$ segment for suitable $p \geq 1$, $\emptyset, (0) \in S_1$ and $\lambda_1 = 1$.

So, in order to estimate $\text{diam}(S(A, x, \alpha))$ pick $f_S, f_R \in S(A, x, \alpha)$. Notice that $S \cap R \neq \emptyset$ (both segments contain the set $\{\emptyset, (1)\}$). However we can find U, V two disjoint segments which are $p - q$ and $p - r$ segments, for suitable $p, q, r \geq 2$ such that

$$S = (S \cap R) \cup U \quad \text{and} \quad R = (S \cap R) \cup V.$$

Then

$$f_R - f_S = f_{S \cap R} + f_V - f_{S \cap R} - f_U = f_V - f_U.$$

By Proposition 4.2 we deduce that

$$\|f_V - f_U\| \leq \frac{5}{3} \Rightarrow \|f_R - f_S\| \leq \frac{5}{3}.$$

From the arbitrariness of $f_R, f_S \in S(A, x, \alpha)$ we deduce that

$$\text{diam}(S(A, x, \alpha)) \leq \frac{5}{3}.$$

Hence

$$\inf_{\alpha > 0} \text{diam}(S(A, x, \alpha)) \leq \frac{5}{3}.$$

Now theorem 1.1 applies and

$$\inf_{\alpha > 0} \text{diam}(S(B_{JH_\infty^*}, x, \alpha)) \leq \frac{5}{3}$$

so we are done. \blacksquare

In particular, the above theorem shows that JH_∞^* fails to have the w^* -slice diameter two property, and so every diameter two property.

In view of the element x in last theorem, it seems that the fact that $\emptyset \in \text{supp}(x)$ is a very important fact (it allowed us to describe easily the elements of $S(A, x, \alpha)$). This fact will become clear in next section.

5. AN HYPERPLANE OF JH_∞^* SATISFYING THE w^* -STRONG DIAMETER TWO PROPERTY.

We will consider T defined as in the previous section. Let

$$N := \left\{ x : T \longrightarrow \mathbb{R} \left/ \begin{array}{l} x \text{ is a finitely nonzero function} \\ x(\emptyset) = 0 \end{array} \right. \right\}.$$

Now consider over N the norm defined in the previous section. In other words

$$\|x\| := \sup \sum_{i=1}^k \left| \sum_{t \in S_i} x(t) \right|,$$

where the sup is taken over all families of admissible segments $\{S_1, \dots, S_k\}$ in T .

Now define M as the completion of N under the above norm.

Note that $i : N \hookrightarrow JH_\infty$ is a linear isometry. So, it can be uniquely extended to a linear isometry $\Phi : M \longrightarrow JH_\infty$ and, as a consequence, M can be view as a closed subspace of JH_∞ .

Remark 5.1. Given $x \in N$, notice that in the definition of the norm we can consider only families of admissible segments which are $p - q$ segments with $p \geq 1$. This is an important fact which will allow us to conclude the w^* -strong diameter two property for M^*

For $S \subseteq T$ a segment define $f_S \in M^*$ by

$$f_S(x) = \sum_{t \in S} x(t) \quad \forall x \in N.$$

The first consequence of the previous Remark is that

$$A := \left\{ \sum_{i=1}^n \lambda_i f_{S_i} \left/ \begin{array}{l} |\lambda_i| = 1 \quad \forall i \in \{1, \dots, n\} \\ \{S_1, \dots, S_n\} \text{ family of admissible segments} \\ \emptyset \notin S_i \quad \forall i \in \{1, \dots, n\} \end{array} \right. \right\}$$

is a norming set in B_{M^*} . Hence

$$(5.1) \quad B_{M^*} = \overline{\text{co}}^{w^*}(A),$$

is an immediate consequence of Hahn-Banach's theorem.

We will use that fact in order to prove that M^* enjoys the w^* -strong diameter two property.

Theorem 5.2. *M^* has the w^* -strong diameter two property.*

Proof. Let $C := \sum_{i=1}^n \lambda_i S(B_{M^*}, x_i, \varepsilon)$ a convex combination of w^* -slices in B_{M^*} , being x_1, \dots, x_n finitely non-zero functions defined on T . Let prove that $\text{diam}(C) = 2$.

To this aim, for each $i \in \{1, \dots, n\}$ we can find $n_i \in \mathbb{N}$, $\{S_1^i, \dots, S_{n_i}^i\}$ a family of admissible segments in T and $\mu_1^i, \dots, \mu_{n_i}^i \in \{-1, 1\}$ such that

$$\sum_{i=1}^n \lambda_i \sum_{j=1}^{n_i} \mu_j^i f_{S_j^i} \in C.$$

Now for every $i \in \{1, \dots, n\}$ we have that S_j^i is a $p_i - q_i$ segment for each $j \in \{1, \dots, n_i\}$. We can assume that $q_1 = q_2 = \dots = q_n = r$ and that $r > \max_{1 \leq i \leq n} p_i$ because x_1, \dots, x_n have finite support and each element on T has infinitely many offspring.

Again due to the finiteness of $\text{supp}(x_i)$ for each $i \in \{1, \dots, n\}$ we can find B a branch in T such that

$$B \cap \left(\bigcup_{i=1}^n \text{supp}(x_i) \right) = \emptyset.$$

For each $i \in \{1, \dots, n\}$ we can choose $S_i \subseteq B$ a $p_i - r$ segment in T . As $S_i \cap \text{supp}(x_i) = \emptyset$ and $\{S_1^i, \dots, S_{n_i}^i, S_i\}$ is a family of admissible segments in T we deduce that

$$\sum_{i=1}^n \lambda_i \left(\sum_{j=1}^{n_i} \mu_j^i f_{S_j^i} \pm f_{S_i} \right) \in C.$$

Hence

$$\begin{aligned} \text{diam}(C) &\geq \left\| \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^{n_i} \mu_j^i f_{S_j^i} + f_{S_i} \right) - \sum_{i=1}^n \lambda_i \left(\sum_{j=1}^{n_i} \mu_j^i f_{S_j^i} - f_{S_i} \right) \right\| = \\ &= 2 \left\| \sum_{i=1}^n \lambda_i f_{S_i} \right\|. \end{aligned}$$

Let's prove that $\|\sum_{i=1}^n \lambda_i f_{S_i}\| = 1$. Remark's that $\|\sum_{i=1}^n \lambda_i f_{S_i}\| \leq 1$ is obvious in view of triangle inequality. Moreover, as S_i is a $p_i - r$ segment in T and $p_i < r \forall i \in \{1, \dots, n\}$ we deduce the existence of $\alpha \in \bigcap_{i=1}^n S_i$. Now $e_\alpha \in S_M$. Hence

$$\left\| \sum_{i=1}^n \lambda_i f_{S_i} \right\| \geq \sum_{i=1}^n \lambda_i f_{S_i}(e_\alpha) = \sum_{i=1}^n \lambda_i = 1.$$

Thus $\text{diam}(C) = 2$ as desired.

■

Last theorem shows that M^* has each w^* diameter two property and so the norm of M is octahedral. Also, it is easy to see that M has the

strong diameter two property, as proved for JH , and so the norm of M^* is octahedral. As M is separable, we deduce the following

Corollary 5.3. *M has the almost Daugavet property.*

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